University of Groningen
Exam Numerical Mathematics 1, June 19, 2015

Use of a simple calculator is allowed. All answers need to be motivated.
In front of the exercises you find its weight. In fact it gives the number of tenths which can be gained in the final mark. In total 5.4 points can be scored with this exam.

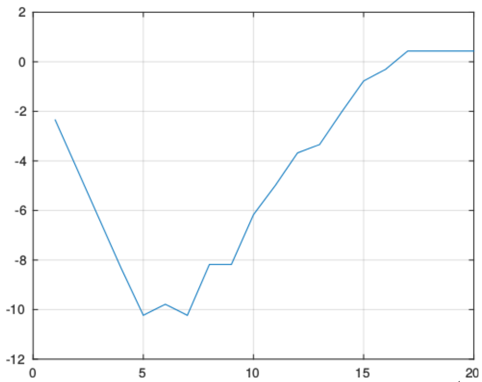
Exercise 1

(a) $\boxed{4}$ Given the data pairs $(x, y) = \{(-1, 0), (-0.5, -1), (0.5, 1), (1, 0)\}$. Give first the general form of the interpolation polynomial expressed in the Lagrange characteristic polynomials and next indicate how it is defined for an interpolation on the given data points.

(b) $\boxed{4}$ The data in part (a) is obtained from the function $\sin(\pi x)$. Give the general form of the interpolation error and next define the one for this special function and data on the interval [-1,1]. Show that it can be bounded by $\max_{y\in[0,1]} |(y-1)(y-1/4)|\pi^4/4! = \frac{1}{4}\pi^4/4!$.

(c) $\boxed{5}$ The plot on the right has been generated by the code on the left.

```
a=1.0
for i=1:20
  h=0.1^i;
  d(i)=(exp(a+h)-exp(a-h))/(2*h)-exp(a);
end
plot(log10(abs(d)))
grid
```



It contains three phases: one downward, one upward, and finally a constant phase (in fact the value is e). Explain what happens in the respective phases. Also explain the value of the slope in the first phase.

(d) $\boxed{2}$ Define both the trapezoidal rule and the composite trapezoidal rule for an integration of a function f over an interval $[a, b]$. The error of the trapezoidal rule is given by $E = -(b-a)^3 f''(\xi)/12$. What is the degree of exactness of this method? Why?

Exercise 2

Consider the linear system $Ax = b$, where A is of order n.

(a) Let $A = LU$ with L a lower and U an upper triangular matrix.

(i) $\boxed{1}$ How can one compute the determinant of A from this LU factorization?

(ii) $\boxed{5}$ Explain that the construction of the LU factorization is a process of n-1 steps in which in each step similar operations are performed. Show from this that if the first step costs $p(n)$ operations that the total amount of operations is $p(n)+p(n-1)+p(n-2)+\cdots+p(2)$. Also give p(n). Given that $\sum_{k=1}^{n} k^\gamma = n^{\gamma+1}/(\gamma+1) + O(n^\gamma)$, where $\gamma = 0, 1, 2, \cdots$, determine the main term in the number of operations to compute the LU factorization.

Continue on other side!

(b) Suppose that A is Symmetric Positive Definite (SPD).

(i) $\boxed{4}$ Let $A = P - C$, when does the iteration $Px^{(k+1)} = Cx^{(k)} + b$, with $x^{(0)}$ given, converge and why? Also indicate the iteration matrix.

(ii) $\boxed{4}$ Suppose the iteration matrix in the previous part has one eigenvalue that is bigger than all others. Where does the difference $x^{(k+1)} - x^{(k)}$ converge to and why?

(iii) $\boxed{1}$ Is there any kind of restriction like in part (a) present if we apply the Conjugate Gradient method to the above linear system with A SPD?

Exercise 3

Consider the nonlinear system $f(x) = 0$, where f is a mapping from R^n to R^n.

(a) $\boxed{4}$ Derive Newton's method for the above system and indicate which linear system has to be solved in each step.

(b) $\boxed{4}$ Suppose $f_1 = x_1(x_1 + x_2 - 1)$, $\quad f_2 = \frac{1}{100} + x_2 \ln(1 + x_2 - x_1)$. Give the Jacobian matrix of f.

(c) $\boxed{1}$ If $n = 1$, one can define a fixed point method by $\phi = x + af(x)$. How? Why does it give the zero of $f(x) = 0$ if it converges?

(d) $\boxed{3}$ Expressed in ϕ, when will the fixed point method show quadratic convergence?

Exercise 4

Consider a system of ODEs

$$\frac{d}{dt}y(t) = f(t, y(t)), \text{ with } y(0) = y_0$$

(a) Consider the method $u_{k+1} = u_{k-1} + 2\Delta t f(t_k, u_k)$.
$\boxed{4}$ Apply the method to the test equation $y' = \lambda y$. Show that one of the roots of the resulting equation is in magnitude always bigger than 1 for $\lambda \Delta t < 0$ and λ real.

(b) Consider on $[0, 1]$ for $u(x, t)$ the diffusion equation $\partial u/\partial t = \partial^2 u/\partial x^2 + x\exp(-t)$ with initial condition $u(x, 0) = \sin(\pi x)$ and boundary conditions $u(0, t) = \sin^2(t)$ and $u(1, t) = 0$. Let the grid in x-direction be given by $x_i = i\Delta x$ where $\Delta x = 1/m$.

(i) $\boxed{4}$ Show that $\frac{\partial^2 u}{\partial x^2}(x_i, t) = \frac{u(x_{i+1},t) - 2u(x_i,t) + u(x_{i-1},t)}{\Delta x^2} + O(\Delta x^2)$.

(ii) $\boxed{3}$ Show that the system of ordinary differential equations (ODEs) that results from using the expression in (i) is of the form

$$\frac{d}{dt}\mathbf{u}(t) = -\frac{1}{\Delta x^2} A\mathbf{u}(t) + \mathbf{b}(t)$$

and give A, $\mathbf{b}(t)$ and $\mathbf{u}(0)$.

(iii) $\boxed{1}$ Given that the eigenvalues of A are in the interval $(0, 4)$. Is it possible to perform a computation with the method introduced in part (a) on an infinite time interval?